The One Four Problem

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Background

The Four Fours problem is a well known mathematical puzzle first published in the late 19^{th} century. The puzzle as original stated asks for mathematical expressions evaluating to each integer from 0 to a certain maximum, often 50 or 100. These expressions are restricted to using only the digit '4', and must have exactly 4 such digits. Additionally, one may use functions such as addition (+), subtraction (-), multiplication ($\cdot$), division ($\div$), concatenation (i.e. 44), and parentheses (i.e. (4+4)*4). Usually, one may also use functions that do not require additional numerals to write, for instance the square root function ($\sqrt{4} = 2$), or the factorial function (4!=24).

Previous Results

Wikipedia provides various ways of constructing any integer n from four fours, including the following simple construction:

$$n = \log_{\sqrt{4}/4} \log_4 \underbrace{\sqrt{\sqrt{\cdots \sqrt{4}}}}_{n}$$

Note that we can easily reduce the number of fours here to three in the following way:

$$n = -\log_{\sqrt{4}} \log_4 \underbrace{\sqrt{\sqrt{\cdots \sqrt{4}}}}_{n}$$

But from here there are no obvious further improvements.

Goals

In this paper, we prove that there exists a mathematical expression evaluating to any integer which uses only a single digit 4, although we will not provide an explicit construction for the family of such expressions.

Proof

Lemma 1. *For all positive integers x, y such that $1 < x < y$, there exists a positive integer m such that*

$$x < \underbrace{\sqrt{\sqrt{\cdots\sqrt{y}}}}_{m-1} \leq x^2$$

Proof. Let us define a sequence of real numbers b where $b_m = \underbrace{\sqrt{\sqrt{\cdots\sqrt{y}}}}_{m-1}$. b has the following properties

(1) $b_m = b_{m+1}^2$
(2) $x < y = b_1$
(3) $\lim_{m\to\infty} b_m = 1$
(4) There exists an m such that $b_m \leq x$

(1) and (2) are trivial, and (3) comes from noting that $b_m = y^{\frac{1}{2^{m-1}}}$, so $\lim_{m\to\infty} b_m = y^0 = 1$. (4) comes as a direct consequence of 3, since by the definition of a limit, there exist infinitely many elements of b that are less than or equal to $1 + \epsilon$ for all $\epsilon > 0$, so taking $\epsilon = 1$ we get there are infinitely many elements of b such that $b_m \leq 2 \leq x$

(2) and (4) gives us that there exists an m such that $b_{m+1} \leq x$ and $x < b_m$. But by (1), $b_m = b_{m+1}^2 \leq x^2$. Therefore, $x < b_m \leq x^2$, so we have proven that there exists an m such that $x < \underbrace{\sqrt{\sqrt{\cdots\sqrt{y}}}}_{m-1} \leq x^2$. □

Lemma 2. *There exists an infinite sequence a_n of positive integers expressible using a single four with $a_1 = 4$ and satisfying the property $a_n < a_{n+1} \leq a_n^2$ for all $n = 1, 2, 3....$*

Proof. Define $a_1 = 4$. Clearly 4 is expressible using only one 4.

Now, we recursively define $a_{n+1} = \left\lceil \underbrace{\sqrt{\sqrt{\cdots\sqrt{a_n!}}}}_{m-1} \right\rceil$ for some positive integer m. We know that $a_n \geq 4$ for all n, so $a_n < a_n!$ for all n. Thus, we can apply Lemma 1 to show that there exists an m such that $a_n < \underbrace{\sqrt{\sqrt{\cdots\sqrt{a_n!}}}}_{m-1} \leq a_n^2$. Taking this m for our construction of a_{n+1}, we can show that $a_n < a_{n+1} \leq a_n^2$, as desired. □

Theorem 3. *Every integer is expressible using a single four.*

Proof. By Lemma 2, we are able to construct a sequence a of positive integers expressible using a single four with $a_1 = 4$ and satisfying $a_n < a_{n+1} \leq a_n^2$ for all $n = 1, 2, 3....$ Now, consider the sequence $b_n = \ln a_n$. We know that $b_1 = \ln 4$, and that

$$a_n < a_{n+1} \leq a_n^2$$

$$\ln a_n < \ln a_{n+1} \leq \ln a_n^2 = 2 \ln a_n$$

$$b_n < b_{n+1} \leq 2b_n$$

Finally, consider the sequence $c_n = \ln b_n$. We know that $c_1 = \ln \ln 4 < 1$ and that

$$b_n < b_{n+1} \leq 2b_n$$

$$\ln b_n < \ln b_{n+1} \leq \ln 2b_n = \ln b_n + \ln 2$$

$$c_n < c_{n+1} \leq c_n + \ln 2 < c_n + 1$$

Furthermore, the sequence a is unbounded, which implies sequence c is also unbounded. Therefore, for all non-negative integers m, there exists an n such that $\lfloor c_n \rfloor = m$, and for any negative integer m, there exists an n such that $-\lfloor c_n \rfloor = m$. □

Limitations and Further Research

One major limitation of this method for constructing integers out of a single four is that it does not provide a closed form expression for generating a particular integer, it merely proves that one exists, and provides a general framework for finding an example. For instance, if we were to look for a construction for the integer 3, we would have to list out values of the sequence a_n as described in Theorem 3 as long as we need:

$a_1 = 4$

$a_2 = \left\lceil \sqrt{4!} \right\rceil = 5$

$a_3 = \left\lceil \sqrt{5!} \right\rceil = \left\lceil \sqrt{\left\lceil \sqrt{4!} \right\rceil !} \right\rceil = 11$

$a_4 = \left\lceil \sqrt{\sqrt{11!}} \right\rceil = \left\lceil \sqrt{\sqrt{\left\lceil \sqrt{\left\lceil \sqrt{4!} \right\rceil !} \right\rceil !}} \right\rceil = 80$

$$a_5 = \left\lceil \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{80!}}}}} \right\rceil = \left\lceil \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\left\lceil \sqrt{\left\lceil \sqrt{4!} \right\rceil !} \right\rceil !}} \right\rceil !}}}}} \right\rceil = 5179$$

And then calculate the natural logarithm of the natural logarithm of these values to find

$$\ln \ln 5179 = \ln \ln \left\lceil \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\left\lceil \sqrt{\left\lceil \sqrt{4!} \right\rceil !} \right\rceil !}} \right\rceil !}}}}} \right\rceil \approx 2.146$$

$$\left\lceil \ln \ln \left\lceil \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\left\lceil \sqrt{\left\lceil \sqrt{4!} \right\rceil !} \right\rceil !}} \right\rceil !}}}}} \right\rceil \right\rceil = 3$$

Clearly, this is an unwieldy process which could not be handled by current computers for any target integers greater than 3.

Another question that arises is whether it is possible to construct any integer from a single four without using any form of logarithm, since it can be considered "cheating" to use the letters in ln or log, as they are not mathematical symbols, and are considered to trivialize the original problem of the four fours. As far as I know, this is an open question, but I have made considerable progress in showing that this can be done.

I hope to release further insights into this question in the near future. Finally, I have provided constructions for integers up to 15 without using logarithms as an appendix for the amusement of the reader on the next page.

1. $\lceil .4 \rceil$

2. $\sqrt{4}$

3. $\left\lceil \sqrt{\left\lceil \sqrt{4!} \right\rceil} \right\rceil$

4. 4

5. $\left\lceil \sqrt{4!} \right\rceil$

6. $\left\lceil \sqrt{\left\lceil \sqrt{4!} \right\rceil} \right\rceil !$

7. $\left\lceil \sqrt{\sqrt{\sqrt{\left\lfloor \sqrt{\left\lceil \sqrt{4!} \right\rceil}\,! \right\rfloor !}}} \right\rceil$

8. $\left\lfloor \sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\sqrt{\left\lfloor \sqrt{\left\lceil \sqrt{4!} \right\rceil}\,! \right\rfloor !}}} \right\rceil !}} \right\rfloor$

9. $\left\lceil \sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\sqrt{\left\lfloor \sqrt{\left\lceil \sqrt{4!} \right\rceil}\,! \right\rfloor !}}} \right\rceil !}} \right\rceil$

10. $\left\lfloor \sqrt{\left\lceil \sqrt{4!} \right\rceil !} \right\rfloor$

11. $\left\lceil \sqrt{\left\lceil \sqrt{4!} \right\rceil !} \right\rceil$

12. $\left\lfloor \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\left\lfloor \sqrt{\sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\left\lfloor \sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\sqrt{\left\lfloor \sqrt{\left\lceil \sqrt{4!} \right\rceil}\,! \right\rfloor !}}} \right\rceil !}} \right\rfloor !}} \right\rceil !}}} \right\rfloor !}}}}} \right\rfloor$

13. $\left\lceil \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\left\lfloor \sqrt{\sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\left\lfloor \sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\sqrt{\left\lfloor \sqrt{\left\lceil \sqrt{4!} \right\rceil}\,! \right\rfloor !}}} \right\rceil !}} \right\rfloor !}} \right\rceil !}}} \right\rfloor !}}}}} \right\rceil$

14. $\left\lfloor \sqrt{\sqrt{\left\lfloor \sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\sqrt{\left\lfloor \sqrt{\left\lceil \sqrt{4!} \right\rceil}\,! \right\rfloor !}}} \right\rceil !}} \right\rfloor !}} \right\rfloor$

15. $\left\lceil \sqrt{\sqrt{\left\lfloor \sqrt{\sqrt{\left\lceil \sqrt{\sqrt{\sqrt{\left\lfloor \sqrt{\left\lceil \sqrt{4!} \right\rceil}\,! \right\rfloor !}}} \right\rceil !}} \right\rfloor !}} \right\rceil$