1 The Duck, Dog, Airplane, and Sun Problem

The original duck and dog problem is somewhat well known and goes roughly as follows:

> You are a duck, and you start off on the very center of a round lake. On the land around the lake there is a dog that wants to catch you, but the dog can't swim. You want to escape your predicament, and the only way for you to do so is to get to land so that you can fly away. Suppose that the dog can run at a certain constant maximum speed. At what ratio of the dog's speed do you have swim to be able to escape?

The naïve approach of simply running toward the end of the lake opposite the dog will result in an answer of π^{-1}, but we can do quite a bit better. Anyway, this is an interesting problem with an interesting solution and isn't terribly difficult to solve.

Here's a problem I came up with years ago, before I had even heard of the duck and dog problem:

> You are a solar powered airplane, and you start off at the North Pole. You always want to remain on the half of the Earth that is being lit by the Sun. Assume that there is no planetary tilt, so that the sun always hits half of the Earth along the lines of longitude. The Earth is, however, rotating at a constant speed so that the locations of dark and light regions get swapped twice per day. If the airplane travels at a constant maximum ground speed, what is the lowest speed that the airplane can travel and be able to make it to the South Pole?

As it turns out, the solution to this problem is almost identical to the solution of the Duck and Dog problem.

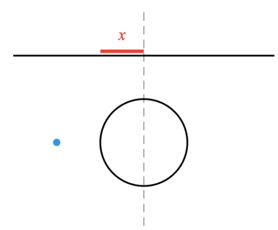
2 The One Four Problem

Another well known problem (actually more of a puzzle), introduced to me by the book *The Man Who Counted* by Malba Tahan, is the Four Fours problem. It asks for ways to construct positive integers using only the digits 4 repeated exactly 4 times, as well as operations that can be written with mathematical symbols. For instance, $1 = 4 - 4 + \frac{4}{4}$, and $2 = 4 - \frac{4+4}{4}$.

That's all in good fun, but we're not really pushing our limits yet. What if we used only three fours? Or two fours? How about just one? I've made some progress, but haven't quite found as nice of a solution as I would have liked.

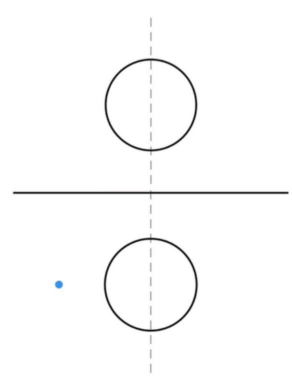
3 FiveThirtyEight's Pinball Wizard meets Inversions

In July 2020, FiveThirtyEight posed the following problem in their weekly Riddler section:

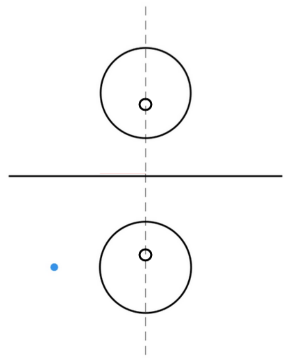
> Riddler Pinball is a game with an infinitely long wall and a circle whose radius is 1 inch and whose center is 2 inches from the wall. The wall and the circle are both fixed and never move. A single pinball starts 2 inches from the wall and 2 inches from the center of the circle.
>
> To play, you flick the pinball toward a spot of your choosing along the wall, specified by its distance x from the point on the wall that's closest to the circle, as shown in the diagram below.
>
> 
>
>
> The goal of the game is simple: Get the ball to bounce as many times as possible.
>
> (Note: This is a geometry problem, not a physics problem. In other words, assume the system is frictionless and that all collisions are perfectly elastic.)

All of the submitted solutions relied heavily on computational power to precisely model where the ball would bounce given an initial target on the wall. Additionally, nobody seemed to be able to rigorously prove that the ball would end up bouncing infinitely, although this is ostensibly the correct answer.

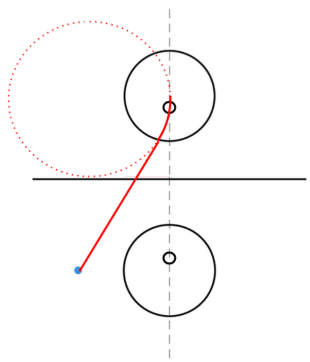
My gut reaction to seeing this problem was to use inversions so that we don't have to deal with the ball changing angles after each bounce. After all, this is how I've solved problems related to halls of mirrors or pool tables before. There's just one problem: I've only ever dealt with the ball bouncing off of lines before, which just requires you to "flip" everything over the axis of the mirror. Unfortunately, we only get this far since things get weird when you start bouncing things off a circle:



For bouncing things off a circle, you get into the realm of hyperbolic geometry. Let's ignore the center line for now and invert the circles onto one another:



And now when we shoot the ball and it bounces off the top circle, the inversion of the path within the top circle will no longer be a straight line. It'll be the circle tangent to the ball's initial path and intersecting the center of the top circle. So something like this...



Or at least, that's what would happen if it didn't hit into the inversion of the inversion of the bottom circle, that is, the small circle inside the top circle, and instead kept going toward infinity, represented by the center of the top circle.

At this point, things start getting quite confusing, probably made more so by the fact that I don't actually know anything about hyperbolic geometry or inversions of circles.

What I do know is that by repeating the process from above, you eventually end up with inversions of inversions of inversions, all piled on top of each other ad infinitum, and the path that the ball takes can be represented by a bunch of little arcs that stray toward the centers of the inversion circles, only to get diverted by the next inversion circle, thus continuing the cycle. Then, if at some point the ball's path doesn't intersect with the next inversion circle, it reaches the center of the circle that it's in, representing the ball flying off in space to infinity, meaning no more bounces.

Now the question is: can this method actually be used to prove something interesting about this problem? Can we use these nested inversions to prove that we can get infinitely many bounces if we aim just right? What about calculating the exact angle in which we need to throw the ball in order to get those infinitely many bounces? I have no solution to this right now so please let me know of any insights...